

Analysis of a method for identification of obstacles

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Abstract

Some difficulties are pointed out in the methods for identification of obstacles based on the numerical verification of the inclusion of a function in the range of an operator. Numerical examples are given to illustrate theoretical conclusions. Alternative methods of identification of obstacles are mentioned: the Support Function Method (SFM) and the Modified Rayleigh Conjecture (MRC) method.

1991 *Mathematics Subject Classification:* Primary 78A46, 65N21, Secondary 35R30.

Key words and phrases: Inverse scattering, obstacle identification, support function method, linear sampling method, Modified Rayleigh Conjecture method.

1 Analysis

During the last decade there are many papers published, in which methods for identification of an obstacle are proposed, which are based on a numerical verification of the inclusion of some function $f := f(\alpha, z)$, $z \in \mathbb{R}^3$, $\alpha \in S^2$, in the range $R(B)$ of a certain operator B . Examples of such methods include [2], [3],[6]. It is proved in this paper that the methods, proposed in the above papers, have essential difficulties. This also is demonstrated by numerical experiments. Although it is true that $f \notin R(B)$ when $z \notin D$, it turns out that in any neighborhood of f , however small, there are elements from $R(B)$. Also, although $f \in R(B)$ when $z \in D$, there are elements in every neighborhood of f ,

however small, which do not belong to $R(B)$ even if $z \in D$. Therefore it is not possible to construct a stable numerical method for identification of D based on checking the inclusions $f \notin R(B)$ and $f \in R(B)$.

We prove below that the range $R(B)$ is dense in the space $L^2(S^2)$.

Assumption (A): *We assume throughout that k^2 is not a Dirichlet eigenvalue of the Laplacian in D .*

Let us introduce some *notations*: $N(B)$ and $R(B)$ are, respectively, the null-space and the range of a linear operator B , $D \in \mathbb{R}^3$ is a bounded domain (obstacle) with a smooth boundary S , $D' = \mathbb{R}^3 \setminus D$, $u_0 = e^{ik\alpha \cdot x}$, $k = \text{const} > 0$, $\alpha \in S^2$ is a unit vector, N is the unit normal to S pointing into D' , $g = g(x, y, k) := g(|x - y|) := \frac{e^{ik|x-y|}}{4\pi|x-y|}$, $f := e^{-ik\alpha' \cdot z}$, where $z \in \mathbb{R}^3$ and $\alpha' \in S^2$, $\alpha' := xr^{-1}$, $r = |x|$, $u = u(x, \alpha, k)$ is the scattering solution:

$$(\Delta + k^2)u = 0 \quad \text{in } D', u|_S = 0, \quad (1)$$

$$u = u_0 + v, \quad v = A(\alpha', \alpha, k)e^{ikr}r^{-1} + o(r^{-1}), \quad \text{as } r \rightarrow \infty, \quad xr^{-1} = \alpha', \quad (2)$$

where $A := A(\alpha', \alpha, k)$ is called the scattering amplitude, corresponding to the obstacle D and the Dirichlet boundary condition. Let $G = G(x, y, k)$ be the resolvent kernel of the Dirichlet Laplacian in D' :

$$(\Delta + k^2)G = -\delta(x - y) \quad \text{in } D', G|_S = 0, \quad (3)$$

and G satisfies the outgoing radiation condition.

If

$$(\Delta + k^2)w = 0 \quad \text{in } D', \quad w|_S = h, \quad (4)$$

and w satisfies the radiation condition, then ([7]) one has

$$w(x) = \int_S G_N(x, s)h(s)ds, \quad w = a(\alpha', k)e^{ikr}r^{-1} + o(r^{-1}), \quad \text{as } r \rightarrow \infty, \quad xr^{-1} = \alpha'. \quad (5)$$

We write $a(\alpha')$ for $a(\alpha', k)$, and

$$a(\alpha') := Bh := \frac{1}{4\pi} \int_S u_N(s, -\alpha')h(s)ds, \quad (6)$$

as follows from Ramm's lemma:

Lemma 1 ([7], p.46) *One has:*

$$G(x, y, k) = g(r)u(y, -\alpha', k) + o(r^{-1}), \quad \text{as } r = |x| \rightarrow \infty, \quad xr^{-1} = \alpha', \quad (7)$$

where u is the scattering solution (1)-(2).

One can write the scattering amplitude as:

$$A(\alpha', \alpha, k) = -\frac{1}{4\pi} \int_S u_N(s, -\alpha')e^{ik\alpha \cdot s}ds. \quad (8)$$

The following claim is proved in [6]:

Claim: $f := e^{-ik\alpha' \cdot z} \in R(B)$ if and only if $z \in D$.

Proof of the claim. Our proof is based on the results in [7].

a) Let us assume that $f = Bh$, i.e., $f \in R(B)$, and prove that $z \in D$. Define $p(y) := g(y, z) - \psi(y)$, where $\psi(y) := \int_S G_N(s, y)h(s)ds$. The function $p(y)$ solves the Helmholtz equation (4) in the region $|y| > |z|$ and $p(y) = o(\frac{1}{|y|})$ as $|y| \rightarrow \infty$ because of (7) and of the relation $Bh = f$. Therefore (see [7], p.25) $p = 0$ in the region $|y| > |z|$. Since ψ is bounded in D' and $g(y, z) \rightarrow \infty$ as $y \rightarrow z$, we get a contradiction unless $z \in D$. Thus, $f \in R(B)$ implies $z \in D$.

b) Let us prove that $z \in D$ implies $f \in R(B)$. Define $\psi(y) := \int_S G_N(s, y)g(s, z)ds$, and $h := g(s, z)$. Then, by Green's formula, one has $\psi(y) = g(y, z)$. Taking here $|y| \rightarrow \infty$, $\frac{y}{|y|} = \alpha'$, and using (7), one gets $f = Bh$, so $f \in R(B)$. The claim is proved. \square

Consider $B : L^2(S) \rightarrow L^2(S^2)$, and $A : L^2(S^2) \rightarrow L^2(S^2)$, where B is defined in (6) and $Aq := \int_{S^2} A(\alpha', \alpha)q(\alpha)d\alpha$.

Theorem 1. *The ranges $R(B)$ and $R(A)$ are dense in $L^2(S^2)$.*

Proof. Recall that assumption (A) holds. It is sufficient to prove that $N(B^*) = \{0\}$ and $N(A^*) = \{0\}$. Assume $0 = B^*q = \int_{S^2} u_N(s, -\alpha')qd\alpha'$, where the overline stands for complex conjugate. Taking complex conjugate and denoting \bar{q} by q again, one gets $0 = \int_{S^2} u_N(s, -\alpha')qda\alpha'$. Define $w(x) := \int_{S^2} u(x, -\alpha')qda\alpha'$. Then $w = w_N = 0$ on S , and w solves equation (1) in D' . By the uniqueness of the solution to the Cauchy problem, $w = 0$ in D' . Let us derive from this that $q = 0$. One has $w = w_0 + V$, where $w_0 := \int_{S^2} e^{-ik\alpha' \cdot x}qda\alpha'$, and $V := \int_{S^2} v(x, -\alpha', k)qda\alpha'$ satisfies the radiation condition. Therefore, $w_0(x) = 0$ in D' , as follows from Lemma 2 proved below. By the unique continuation, $w_0(x) = 0$ in \mathbb{R}^3 , and this implies $q = 0$ by the injectivity of the Fourier transform. This proves the first statement of Theorem 1. Its second statement is proved below. \square

Let us now prove Lemma 2, mentioned above.

We keep the notations used in the above proof.

Lemma 2. If $w = w_0 + V = 0$ in D' , then $w_0 = 0$ in D' .

Proof. The idea of the proof is simple: since w_0 does not satisfy the radiation condition, and V satisfies it, one concludes that $w_0 = 0$. Let us give the details. The key formula is ([7], p.54):

$$\int_{S^2} e^{ik\alpha \cdot \beta r} q(\beta) d\beta = \frac{2\pi i}{k} [\bar{\gamma}q(-\alpha) - \gamma q(\alpha)] + o\left(\frac{1}{r}\right), \quad r \rightarrow \infty, \quad (9)$$

where $\gamma := e^{ikr}/r$, and one assumes $q \in C^1(S^2)$.

If $r := |x| \rightarrow \infty$, then, by Lemma 2, assuming $q \in C^1(S^2)$, and using the relation $w = w_0 + V = 0$ in D' , one gets $q(\alpha) = 0$ for all $\alpha \in S^2$. Thus, Lemma 2 is proved under the additional assumption $q \in C^1(S^2)$. If $q \in L^2(S^2)$, then one uses a similar argument in a weak sense, i.e., with $x := r\beta$, one considers the inner product in $L^2(S^2)$ of $w_0(r\beta)$ and a smooth test function $h \in C^\infty(S^2)$, and applies Lemma 2 to the function

$\int_{S^2} e^{-ik\alpha' \cdot \beta r} h d\beta$. Then, using arbitrariness of h , one concludes that $q = 0$ as an element of $L^2(S^2)$. Lemma 2 is proved. \square .

Let us prove the second statement of Theorem 1.

Assume now that $A^*q = 0$. Taking complex conjugate, and using the reciprocity relation: $A(\alpha, \beta) = A(-\beta, -\alpha)$, one gets an equation:

$$\int_{S^2} A(\alpha, \beta) h d\beta = 0, \quad (10)$$

where $h = \overline{q(-\beta)}$. Define $w(x) := \int_{S^2} u(x, \beta) h d\beta$. Then $w = w_0 + V$, where $w_0 := \int_{S^2} e^{ik\beta \cdot x} h d\beta$, and $V := \int_{S^2} v(x, \beta) h d\beta$ satisfies the radiation condition. Equation (10) implies that $V = o(\frac{1}{r})$ as $r \rightarrow \infty$. Since function V solves equation (1) and $V = o(\frac{1}{r})$, one concludes (see [7], p.25), that $V = 0$ in D' , so that $w = w_0$ in D' . Thus, $w_0|_S = w|_S = 0$. Since w_0 solves equation (1) in D and $w_0|_S = 0$, one gets, using Assumption (A), that $w_0 = 0$ in D . This and the unique continuation property imply $w_0 = 0$ in \mathbb{R}^3 . Consequently, $h = 0$, so $q = 0$, as claimed. Theorem 1 is proved. \square

Remark 1. In [2] the 2D inverse obstacle scattering problem is considered. It is proposed to solve the equation (1.9) in [2]:

$$\int_{S^1} A(\alpha, \beta) \mathcal{G} d\beta = e^{-ik\alpha \cdot z}, \quad (11)$$

where A is the scattering amplitude at a fixed $k > 0$, S^1 is the unit circle, and z is a point on \mathbb{R}^2 . If $\mathcal{G} = \mathcal{G}(\beta, z)$ is found, the boundary S of the obstacle is to be found by finding those z for which $\|\mathcal{G}\| := \|\mathcal{G}(\beta, z)\|_{L^2(S^1)}$ is maximal. Assuming that k^2 is not a Dirichlet or Neumann eigenvalue of the Laplacian in D , that D is a smooth, bounded, simply connected domain, the authors state Theorem 2.1 [2], p.386, which says that for every $\epsilon > 0$ there exists a function $\mathcal{G} \in L^2(S^1)$, such that

$$\lim_{z \rightarrow S} \|\mathcal{G}(\beta, z)\| = \infty, \quad (12)$$

and (see [2], p.386),

$$\left\| \int_{S^1} A(\alpha, \beta) \mathcal{G} d\beta - e^{-ik\alpha \cdot z} \right\| < \epsilon. \quad (13)$$

There are several questions concerning the proposed method.

First, equation (11), in general, is not solvable. The authors propose to solve it approximately, by a regularization method. The regularization method applies for stable solution of solvable ill-posed equations (with exact or noisy data). If equation (11) is not solvable, it is not clear what numerical "solution" one seeks by a regularization method.

Secondly, since the kernel of the integral operator in (11) is smooth, one can always find, for any $z \in \mathbb{R}^2$, infinitely many \mathcal{G} with arbitrary large $\|\mathcal{G}\|$, such that (13) holds. Therefore it is not clear how and why, using (12), one can find S numerically by the proposed method.

Remark 2. In [2], p.386, Theorem 2.1, it is claimed that for every $\epsilon > 0$ and every $y_0 \in D$ there exists a function \mathcal{G} such that inequality (13) (which is (2.8) on p.386 of [2]) holds and $\|\mathcal{G}\| \rightarrow \infty$ as $y_0 \rightarrow \partial D$. Such a \mathcal{G} is used in [2] in a "simple method for solving inverse scattering problem". However, in fact there exist infinitely many \mathcal{G} such that inequality (13) holds and $\|\mathcal{G}\| \rightarrow \infty$, regardless of where y_0 is. Therefore it is not clear how one can use the method proposed in [2] for solving the inverse scattering problem with any degree of confidence in the result.

Remark 3. In [1] it is mentioned that the methods (called LSM-linear sampling methods) proposed in papers [3], [2], [6] produce numerically results which are inferior to these obtained by linearized Born-type inversion. There is no guarantee of any accuracy in recovery of the obstacle by LSM. Therefore it is of interest to experiment numerically with other inversion methods. In [7], p.94, (see also [8],[11],[12]) a method (SFM-support function method) is proposed for recovery of strictly convex obstacles from the scattering amplitude. This method allows one to recover the support function of the obstacle, and the boundary of the obstacle is obtained from this function explicitly. Error estimates of this method are obtained for the case when the data are noisy [7], p.104. The method is asymptotically exact for large wavenumbers, but it works numerically even for $ka \sim 1$, as shown in [5]. For the Dirichlet, Neumann and Robin boundary conditions this method allows one to recover the support function without a priori knowledge of the boundary condition. If the obstacle is not convex, the method recovers the convex hull of the obstacle. Numerically one can recover the obstacle, after its convex hull is found, by using Modified Rayleigh conjecture method, introduced in [10], or by a parameter-fitting method.

In [13] there is a formula for finding an acoustically soft obstacle from the fixed-frequency scattering data. It is an open problem to develop an algorithm based on this formula.

A numerical implementation of the Linear Sampling Method (LSM), suggested in [2], consists of solving a discretized version of

$$\int_{S^1} A(\alpha, \beta) \mathcal{G} d\beta = e^{-ik\alpha \cdot z}, \quad (14)$$

where A is the scattering amplitude at a fixed $k > 0$, S^1 is the unit circle, $\alpha \in S^1$, and z is a point on \mathbb{R}^2 .

Let $F = \{A\alpha_i, \beta_j\}$, $i = 1, \dots, N$, $j = 1, \dots, N$ be the square matrix formed by the measurements of the scattering amplitude for N incoming, and N outgoing directions. then the discretized version of (14) is

$$F\mathbf{g} = \mathbf{f}, \quad (15)$$

where the vector \mathbf{f} is formed by

$$\mathbf{f}_n = \frac{e^{i\frac{\pi}{4}}}{\sqrt{8\pi k}} e^{-ik\alpha_n \cdot z}, \quad n = 1, \dots, N, \quad (16)$$

see [1] for details.

Denote the Singular Value Decomposition of the far field operator by $F = USV^H$. Let s_n be the singular values of F , $\rho = U^H \mathbf{f}$, and $\mu = V^H \mathbf{f}$. Then the norm of the sought function g is given by

$$\|\mathcal{G}\|^2 = \sum_{n=1}^N \frac{|\rho_n|^2}{s_n^2}. \quad (17)$$

A different LSM is suggested by A. Kirsch in [6]. In it one solves

$$(F^* F)^{1/4} \mathbf{g} = \mathbf{f} \quad (18)$$

instead of (15). The corresponding expression for the norm of \mathcal{G} is

$$\|\mathcal{G}\|^2 = \sum_{n=1}^N \frac{|\mu_n|^2}{s_n}. \quad (19)$$

A detailed numerical comparison of the two LSMs and the linearized tomographic inverse scattering is given in [1].

The conclusions of [1], as well as of our own numerical experiments are that the method of Kirsch (19) gives a better, but a comparable identification, than (17). The identification is significantly deteriorating if the scattering amplitude is available only for a limited aperture, or if the data are corrupted by noise. Also, the points with the *smallest* values of the $\|\mathcal{G}\|$ are the best in locating the inclusion, and not the *largest* one, as required by the theory in [6] and in [2]. In Figures 1 and 2 the implementation of the Colton-Kirsch LSM (17) is denoted by *gnck*, and of the Kirsch method (19) by *gnk*. The Figures show a contour plot of the logarithm of the $\|\mathcal{G}\|$. The original obstacle consisted of two circles of radius 1.0 centered at the points $(-d, 0.0)$ and $(d, 0.0)$. The results of the identification for $d = 2.0$ are shown in Figure 1, and the results for $d = 1.5$ are shown in Figure 2. Note that the actual radius of the circles is 1.0, but it cannot be seen from the LSM identification. Also, one cannot determine the separation between the circles, nor their shapes. Still, the methods are fast, they locate the obstacles, and do not require any knowledge of the boundary conditions on the obstacle. The Support Function Method ([5], [7]) showed a better identification for the convex parts of obstacles. Its generalization for unknown boundary conditions is discussed in [14]. The LSM identification was performed for the scattering amplitude of the obstacles computed by the Boundary Integral Equations method, see [4]. No noise was added to the synthetic data. In all the experiments we used $k = 1.0$, and $N = 60$.

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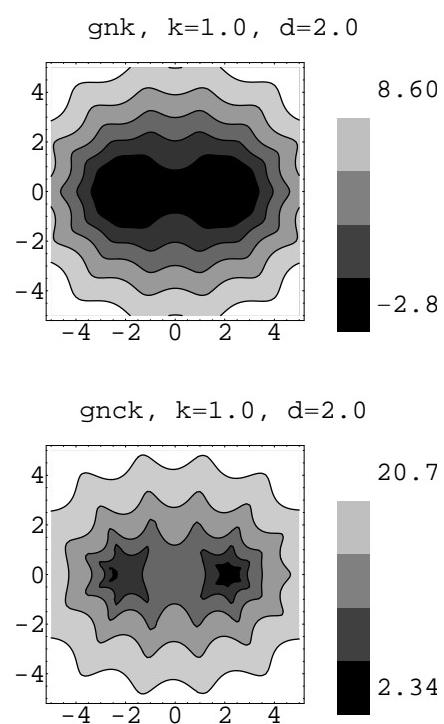


Figure 1: Identification of two circles of radius 1.0 centered at $(-d, 0.0)$ and $(d, 0.0)$ for $d = 2.0$.

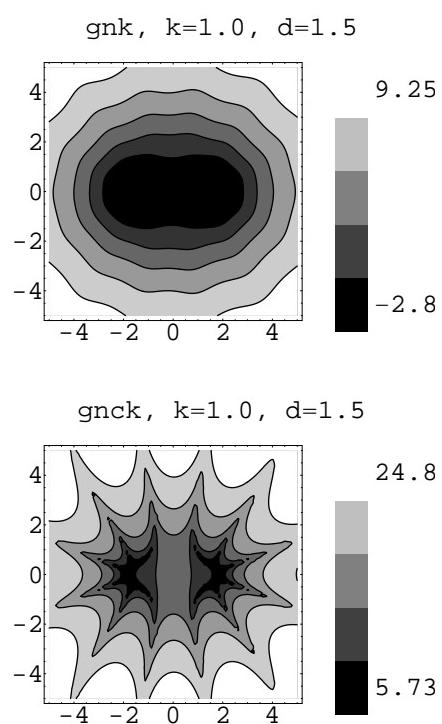


Figure 2: Identification of two circles of radius 1.0 centered at $(-d, 0.0)$ and $(d, 0.0)$ for $d = 1.5$.

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